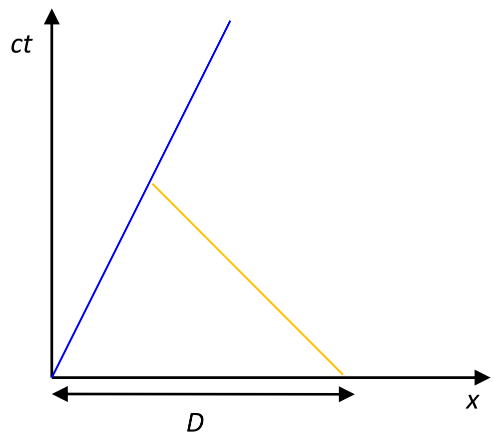
Teacher notes
Topic A

An instructive relativity problem.

The blue line is the worldline of a rocket moving past earth with speed *v*. A laser beam is emitted from *x* = *D* at *t* = 0. When does the beam get to the rocket according to rocket clocks?



Consider the events:

E_1 = beam is emitted

E_2 = beam arrives at rocket

For earth, the time between these events is Δt and $\Delta x = v\Delta t$ since the rocket has moved closer to the launch point in the time Δt. Hence

$$\begin{aligned}\Delta t' &= \gamma(\Delta t - \frac{v}{c^2}\Delta x)\\ &= \gamma(\Delta t - \frac{v}{c^2}v\Delta t) = \gamma\Delta t(1-\frac{v^2}{c^2})\\ &= \gamma\Delta t\frac{1}{\gamma^2} = \frac{\Delta t}{\gamma}\end{aligned}$$

According to earth $\Delta t = \dfrac{D}{v+c}$. This is because the distance between the launch point and the rocket is decreasing at a rate $v+c$. This does not violate the speed of light being the maximum possible. No material body is moving at this speed. Hence $\Delta t' = \dfrac{1}{\gamma}\dfrac{D}{v+c}$.

We can also address this with a spacetime diagram. The equation of the blue line in the diagram above is

$$t = \frac{x}{v}$$

The equation of the photon worldline (orange line) is

$$t = -\frac{(x - D)}{c}$$

The lines intersect at

$$\frac{x}{v} = -\frac{(x - D)}{c}$$
$$cx = -vx + vD$$
$$x = \frac{vD}{c + v}$$

Hence

$$t = \frac{x}{v} = \frac{D}{c + v}$$

just as we found before.

Hence

$$\begin{aligned} t' &= \gamma(t - \frac{v}{c^2}x) \\ &= \gamma(t - \frac{v}{c^2}vt) = \gamma t(1 - \frac{v^2}{c^2}) \\ &= \gamma t\frac{1}{\gamma^2} = \frac{t}{\gamma} = \frac{1}{\gamma}\frac{D}{c+v} \end{aligned}$$

(Playing mathematical games notice, for what it is worth, that

$$t' = \frac{1}{\gamma}\frac{D}{c+v} = \sqrt{1 - \frac{v^2}{c^2}}\frac{D}{c}\frac{1}{1 + \frac{v}{c}} = \frac{D}{c}\sqrt{(1 + \frac{v}{c})(1 - \frac{v}{c})}\frac{1}{1 + \frac{v}{c}} = \frac{D}{c}\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\ .)$$